

LABELED CAUSETS IN DISCRETE QUANTUM GRAVITY

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Abstract

We point out that labeled causets have a much simpler structure than unlabeled causets. For example, labeled causets can be uniquely specified by a sequence of integers. Moreover, each labeled causet processes a unique predecessor and hence has a unique history. Our main result shows that an arbitrary quantum sequential growth process (QSGP) on the set of labeled causets “compresses” in a natural way onto a QSGP on the set of unlabeled causets. The price we have to pay is that this procedure causes an “explosion” of values due to multiplicities. We also observe that this procedure is not reversible. This indicates that although many QSGPs on the set of unlabeled causets can be constructed using this method, not all can, so it is not completely general. We close by showing that a natural metric can be defined on labeled and unlabeled causets and on their paths.

1 Introduction

In the causal set approach to discrete quantum gravity, a causal set (causet) represents a possible universe at a certain time instant and a possible “completed” universe is represented by a path of growing causets [2, 5, 6, 8, 9]. Just as covariance dictates that the laws of physics are independent of the coordinate system employed, in the discrete theory, covariance implies that

order isomorphic causets should be identified. That is, a cuset should be independent of labeling. This is unfortunate because it is very convenient to work with labeled causets. At a fundamental level, the labeling specifies the “birth order” of the vertices of a cuset. Although covariance dictates that a cuset should be independent of the birth order of its vertices, this order is useful in keeping track of the causets. In particular, the labeled offspring of a labeled cuset possess a natural lexicographic total order. This lexicographic order together with its level uniquely specify a labeled cuset in terms of a sequence of positive integers.

Although there are many more labeled causets than unlabeled causets, their graph structure is much simpler. This is because the graph of labeled causets forms a tree which implies that each labeled cuset has a unique producer (predecessor) and hence possesses a unique history. This unique history structure makes it much simpler to construct a candidate quantum dynamics to form a quantum sequential growth process (QSGP) which is the basis for this approach to discrete quantum gravity.

Even though the graphic structure $(\mathcal{P}, \rightarrow)$ of unlabeled causets is quite complicated, the graphic structure $(\mathcal{P}', \rightarrow)$ of labeled causets is simple. A QSGP consists of a sequence $\{\rho_n\}$ of Hilbert space operators, and it is more straightforward to construct these operators for \mathcal{P}' than for \mathcal{P} . The main point of this paper is that even though the QSGP $\{\rho_n\}$ for \mathcal{P}' can be arbitrary and need not satisfy any covariance conditions, we can “compress” $\{\rho_n\}$ to form a QSGP $\{\hat{\rho}_n\}$ on \mathcal{P} that is automatically covariant. We thus obtain the surprising result that any quantum dynamics on \mathcal{P}' compresses in a natural way to a quantum dynamics on \mathcal{P} .

As one might suspect, there is a price to be paid for this fortunate circumstance. Since there are usually many ways to label an unlabeled cuset, the compression map is many-to-one which results in a multiplicity factor. This factor increases with n and may affect the convergence of quantum measures. In this way, events in \mathcal{P}' with finite quantum measure may have corresponding events in \mathcal{P} with infinite quantum measure. We also observe that this procedure is not reversible. That is, a QSGP on \mathcal{P} may not be “expandable” to a QSGP on \mathcal{P}' . This indicates that although many QSGPs on \mathcal{P} can be constructed using this method, not all can, so it is not completely general. In the last section of this article we show that a natural metric can be defined on labeled and unlabeled causets and on their paths.

2 Quantum Sequential Growth Processes

A finite partially ordered set is called a *causet*. In this section we treat only unlabeled causets and two isomorphic causets are considered to be identical. Let \mathcal{P}_n be the collection of all causets of cardinality n , $n = 1, 2, \dots$, and let $\mathcal{P} = \cup \mathcal{P}_n$ be the collection of all causets. If a, b are elements of a causet x , we interpret the order $a < b$ as meaning that b is in the causal future of a . If $a < b$ and there is no c with $a < c < b$, then a is a *parent* of b and b is a *child* of a . An element $a \in x$ for $x \in \mathcal{P}$ is *maximal* if there is no $b \in x$ with $a < b$. If $x \in \mathcal{P}_n$, $y \in \mathcal{P}_{n+1}$ then x *produces* y if y is obtained from x by adjoining a single maximal element a to x . We then write $x \rightarrow y$ and $y = x \uparrow a$. If $x \rightarrow y$, we say that x is a *producer* of y and y is an *offspring* of x . Of course, x may produce many offspring and a causet may be the offspring of many producers.

The transitive closure of \rightarrow makes \mathcal{P} into a partially ordered set itself and we call $(\mathcal{P}, \rightarrow)$ the *causet growth process* (CGP). A *path* in \mathcal{P} is a sequence (string) $\omega_1 \omega_2 \dots$, where $\omega_i \in \mathcal{P}$ and $\omega_i \rightarrow \omega_{i+1}$, $i = 1, 2, \dots$. An *n-path* in \mathcal{P} is a finite string $\omega_1 \omega_2 \dots \omega_n$ where, again $\omega_i \in \mathcal{P}_i$ and $\omega_i \rightarrow \omega_{i+1}$. We denote the set of paths by Ω and the set of n -paths by Ω_n . If $\omega = \omega_1 \omega_2 \dots \omega_n \in \Omega_n$, we define $(\omega \rightarrow) \subseteq \Omega_{n+1}$ by

$$(\omega \rightarrow) = \{\omega_1 \omega_2 \dots \omega_n \omega_{n+1} : \omega_n \rightarrow \omega_{n+1}\}$$

Thus, $(\omega \rightarrow)$ is the set of one-step continuations of ω . If $A \subseteq \Omega_n$ we define $(A \rightarrow) \subseteq \Omega_{n+1}$ by

$$(A \rightarrow) = \cup \{(\omega \rightarrow) : \omega \in A\}$$

The set of all paths beginning with $\omega \in \Omega_n$ is called an *elementary cylinder set* and is denoted by $\text{cyl}(\omega)$. If $A \subseteq \Omega_n$, then the *cylinder set* $\text{cyl}(A)$ is defined by

$$\text{cyl}(A) = \cup \{\text{cyl}(\omega) : \omega \in A\}$$

Using the notation

$$\mathcal{C}(\Omega_n) = \{\text{cyl}(A) : A \subseteq \Omega_n\}$$

we see that

$$\mathcal{C}(\Omega_1) \subseteq \mathcal{C}(\Omega_2) \subseteq \dots$$

is an increasing sequence of subalgebras of the *cylinder algebra* $\mathcal{C}(\Omega) = \cup \mathcal{C}(\Omega_n)$. Letting \mathcal{A} be the σ -algebra generated by $\mathcal{C}(\Omega)$, we have that (Ω, \mathcal{A})

is a measurable space. For $A \subseteq \Omega$, we define the sets $A^n \subseteq \Omega_n$ by

$$A^n = \{\omega_1\omega_2 \cdots \omega_n : \omega_1\omega_2 \cdots \omega_n\omega_{n+1} \cdots \in A\}$$

That is, A^n is the set of n -paths that can be continued to a path in A . We think of A^n as the n -step approximation to A . We have that

$$\text{cyl}(A^1) \subseteq \text{cyl}(A^2) \subseteq \cdots \subseteq A$$

so that $A \subseteq \cap \text{cyl}(A^n)$ but $A \neq \cap \text{cyl}(A^n)$ in general, even if $A \in \mathcal{A}$.

Let $H_n = L_2(\Omega_n)$ be the n -path Hilbert space \mathbb{C}^{Ω_n} with the usual inner product

$$\langle f, g \rangle = \sum \left\{ \overline{f(\omega)} g(\omega) : \omega \in \Omega_n \right\}$$

For $A \subseteq \Omega_n$ the characteristic function $\chi_A \in H_n$ with $\|\chi_A\| = \sqrt{|A|}$ where $|A|$ denotes the cardinality of A . In particular, $1_n = \chi_{\Omega_n}$ satisfies $\|1_n\| = \sqrt{|\Omega_n|}$. A positive operator ρ on H_n that satisfies $\langle \rho 1_n, 1_n \rangle = 1$ is called a *probability operator*. Corresponding to a probability operator ρ we define the *decoherence functional* $D_\rho : 2^{\Omega_n} \times 2^{\Omega_n} \rightarrow \mathbb{C}$ by

$$D_\rho(A, B) = \langle \rho \chi_B, \chi_A \rangle$$

We interpret $D_\rho(A, B)$ as a measure of the interference between the events A, B when the system is described by ρ . We also define the q -measure $\mu_\rho : 2^{\Omega_n} \rightarrow \mathbb{R}^+$ by $\mu_\rho(A) = D_\rho(A, A)$ and interpret $\mu_\rho(A)$ as the quantum propensity of the event $A \subseteq \Omega_n$. In general, μ_ρ is not additive on 2^{Ω_n} so μ_ρ is not a measure. However, μ_ρ is *grade-2 additive* [1, 2, 5, 7] in the sense that if $A, B, C \in 2^{\Omega_n}$ are mutually disjoint, then

$$\mu_\rho(A \cup B \cup C) = \mu_\rho(A \cup B) + \mu_\rho(A \cup C) + \mu_\rho(B \cup C) - \mu_\rho(A) - \mu_\rho(B) - \mu_\rho(C)$$

Let ρ_n be a probability operator on H_n , $n = 1, 2, \dots$. We say that the sequence $\{\rho_n\}$ is *consistent* if

$$D_{\rho_{n+1}}(A \rightarrow, B \rightarrow) = D_{\rho_n}(A, B)$$

for every $A, B \subseteq \Omega_n$. We call a consistent sequence $\{\rho_n\}$ a *quantum sequential growth process* (QSGP). Now let $\{\rho_n\}$ be a QSGP and denote the corresponding q -measures by μ_n . A set $A \in \mathcal{A}$ is *suitable* if $\lim \mu_n(A^n)$ exists (and is finite) in which case we define $\mu(A) = \lim \mu_n(A^n)$. We denote the collection

of suitable sets by $\mathcal{S}(\Omega)$. It follows from consistency that if $A = \text{cyl}(B)$ for $B \subseteq \Omega_n$, then $\lim \mu_m(A^m) = \mu_n(B)$. Hence, $A \in \mathcal{S}(\Omega)$ and $\mu(A) = \mu_n(B)$. We conclude that $\mathcal{C}(\Omega) \subseteq \mathcal{S}(\Omega) \subseteq \mathcal{A}$ and it can be shown that the inclusions are proper, in general. In a certain sense, μ is a q -measure on $\mathcal{S}(\Omega)$ that extends the q -measures μ_n . There are physically relevant sets in \mathcal{A} that are not in $\mathcal{C}(\Omega)$. In this case it is important to know whether such a set A is in $\mathcal{S}(\Omega)$ and to find $\mu(A)$. For example, if $\omega \in \Omega$, then $\{\omega\} = \cap \{\omega\}^n \in \mathcal{A}$ but $\{\omega\} \notin \mathcal{C}(\Omega)$. Also, the complement $\{\omega\}' \notin \mathcal{C}(\Omega)$.

We now consider a method for constructing a QSGP. A *transition amplitude* is a map $\tilde{a}: \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{C}$ such that $\tilde{a}(x, y) = 0$ if $x \not\rightarrow y$ and $\sum_y \tilde{a}(x, y) = 1$ for every $x \in \mathcal{P}$. This is similar to a Markov chain except $\tilde{a}(x, y)$ may be complex. The *amplitude process* (AP) corresponding to \tilde{a} is given by the maps $a_n: \Omega_n \rightarrow \mathbb{C}$ where

$$a_n(\omega_1 \omega_2 \cdots \omega_n) = \tilde{a}(\omega_1, \omega_2) \tilde{a}(\omega_2, \omega_3) \cdots \tilde{a}(\omega_{n-1}, \omega_n)$$

We can consider a_n to be a vector in H_n . Notice that

$$\langle 1_n, a_n \rangle = \sum_{\omega \in \Omega_n} a_n(\omega) = 1$$

Define the rank 1 positive operator $\rho_n = |a_n\rangle\langle a_n|$ on H_n . Since

$$\langle \rho_n 1_n, 1_n \rangle = |\langle 1_n, a_n \rangle|^2 = 1$$

we conclude that ρ_n is a probability operator.

The corresponding decoherence functional becomes

$$\begin{aligned} D_n(A, B) &= \langle \rho_n \chi_B, \chi_A \rangle = \langle \chi_B, a_n \rangle \langle a_n, \chi_A \rangle \\ &= \sum_{\omega \in A} \overline{a_n(\omega)} \sum_{\omega \in B} a_n(\omega) \end{aligned}$$

In particular, for $\omega, \omega' \in \Omega_n$, $D_n(\omega, \omega') = \overline{a_n(\omega)} a_n(\omega')$ are the matrix elements of ρ_n . The q -measure $\mu_n: 2^{\Omega_n} \rightarrow \mathbb{R}^+$ becomes

$$\mu_n(A) = D_n(A, A) = \left| \sum_{\omega \in A} a_n(\omega) \right|^2$$

It is shown in [4] that the sequence $\{\rho_n\}$ is consistent and hence forms a QSGP.

3 Labeled Causets

A *labeling* for a cuset x is a bijection $\ell: x \rightarrow \{1, 2, \dots, |x|\}$ such that $a, b \in x$ with $a < b$ implies that $\ell(a) < \ell(b)$. A *labeled cuset* is a pair (x, ℓ) where ℓ is a labeling of x . For simplicity, we frequently write $x = (x, \ell)$ and call x an ℓ -cuset. Two ℓ -causets x and y are *isomorphic* if there exists a bijection $\phi: x \rightarrow y$ such that $a < b$ if and only if $\phi(a) < \phi(b)$ and $\ell[\phi(a)] = \ell(a)$ for every $a \in x$. Isomorphic ℓ -causets are identified. A given unlabeled cuset x can always be labeled. Just take a maximal element $a \in x$ and label it $\ell(a) = |x|$. Remove a from x to form $x \setminus \{a\}$ and label a maximal element $b \in x \setminus \{a\}$ by $\ell(b) = |x| - 1$. Continue this process until there is only one element c left and label it $\ell(c) = 1$. To show that ℓ is a labeling of x , suppose $d, e \in x$ with $d < e$. Now there is a maximal chain $d < d_1 < \dots < d_k < e$ in x . By the way ℓ was contracted, we have

$$\ell(d) < \ell(d_1) < \dots < \ell(d_k) < \ell(e)$$

so $\ell(d) < \ell(e)$. Whenever, there is a choice of maximal elements we may obtain a new labeling, so there usually are many ways to label a cuset. (There are exceptions, like a chain or antichain.)

We denote the set of ℓ -causets with cardinality n by \mathcal{P}'_n and the set of all ℓ -causets by $\mathcal{P}' = \cup \mathcal{P}'_n$. The definitions of Section 2 such as producer, offspring, paths, QSGPs and APs are essentially the same for ℓ -causets as they were for causets. For $x, y \in \mathcal{P}'$ if $y = x \uparrow a$, we always label a with the integer $|x| + 1 = |y|$. We denote the collection of n -paths in \mathcal{P}' by Ω'_n and the collection of paths in \mathcal{P}' by Ω' . We define the Hilbert spaces $H'_n = L_2(\Omega'_n)$ as before. If $\omega = \omega_1 \omega_2 \dots \omega_n \in \Omega'_n$ we say that ω_j is *contained* in ω , $j = 1, 2, \dots, n$.

Lemma 3.1. *An ℓ -cuset y cannot be the offspring of two distinct ℓ -cuset producers.*

Proof. If we delete the element of y labeled $|y|$ we obtain a producer of y . But any producer of y is obtained in this way so there is only one ℓ -cuset that produces y . \square

The next lemma shows that unlike in \mathcal{P} , paths in \mathcal{P}' never cross (except at ω_1).

Lemma 3.2. *If $x \in \mathcal{P}'_n$, then x is contained in a unique n -path.*

Proof. Let $\omega_n = x$. If we delete the element of ω_n labeled n , then the resulting set ω_{n-1} is an ℓ -causet with $\omega_{n-1} \rightarrow \omega_n$. If we next delete the element of ω_{n-1} labeled $n-1$, then the resulting set ω_{n-2} is an ℓ -causet with $\omega_{n-2} \rightarrow \omega_{n-1}$. Continue this process until we obtain the one element ℓ -causet ω_1 . Then $\omega = \omega_1 \omega_2 \cdots \omega_n$ is an n -path containing x . If there were another n -path $\omega' = \omega'_1 \omega'_2 \cdots \omega'_n$ with $\omega'_n = x$, then $\omega'_{n-1} = \omega_{n-1}$ because of Lemma 3.1. But by Lemma 3.1 again, $\omega'_{n-2} = \omega_{n-2}$ and continuing we obtain $\omega' = \omega$. Hence, ω is unique. \square

Let $x = \{a_1, a_2, \dots, a_n\}$ be an ℓ -causet where we can assume without loss of generality that the label on a_j is j , $j = 1, \dots, n$. Define

$$j_x \uparrow = \{i \in \mathbb{N} : a_j \leq a_i\}$$

Lemma 3.3. *If $x, y, z \in \mathcal{P}'$ and $x \rightarrow y, z$, then $j_y \uparrow \subseteq j_z \uparrow$ or $j_x \uparrow \subseteq j_y \uparrow$ for all $j = 1, 2, \dots, |y|$.*

Proof. Let $y = \{a_1, \dots, a_n\}$ and $z = \{b_1, \dots, b_n\}$. We can assume that $a_i = b_i$, $i = 1, \dots, n-1$. If $a_j \not\leq a_n$ and $b_j \not\leq b_n$, then $j_y \uparrow = j_z \uparrow$. If $a_j \leq a_n$ and $b_j \leq b_n$, then again $j_y \uparrow = j_z \uparrow$. If $a_j \leq a_n$ and $b_j \not\leq b_n$ then $j_z \uparrow \subseteq j_y \uparrow$ and if $a_j \not\leq a_n$ and $b_j \leq b_n$ then $j_y \uparrow \subseteq j_z \uparrow$. \square

Order the offspring of $x \in \mathcal{P}'$ lexicographically as follows. If $x \rightarrow y, z$ then $y < z$ if $1_y \uparrow = 1_z \uparrow, \dots, j_y \uparrow = j_z \uparrow, (j+1)_y \uparrow \subsetneq (j+1)_z \uparrow$.

Theorem 3.4. *The relation $<$ is a total order.*

Proof. Clearly $y \not< y$. If $y < z$, then $z \not< y$. Suppose that $x \rightarrow y, z, u$ and $y < z, z < u$. Then $1_y \uparrow = 1_z \uparrow, \dots, j_y \uparrow = j_z \uparrow, (j+1)_y \uparrow \subsetneq (j+1)_z \uparrow$ and $1_z \uparrow = 1_u \uparrow, \dots, k_z \uparrow = k_u \uparrow, (k+1)_z \uparrow \subsetneq (k+1)_u \uparrow$. We then have

$$1_y \uparrow = 1_u \uparrow, \dots, \min(j, k)_y \uparrow = \min(j, k)_u \uparrow$$

If $\min(j, k) = j$, then

$$1_y \uparrow = 1_u \uparrow, \dots, j_y \uparrow = j_z \uparrow = j_u \uparrow, (j+1)_y \uparrow \subsetneq (j+1)_z \uparrow = (j+1)_u \uparrow$$

If $\min(j, k) = k$, then

$$1_y \uparrow = 1_u \uparrow, \dots, k_y \uparrow = k_z \uparrow = k_u \uparrow, (k+1)_y \uparrow = (k+1)_z \uparrow \subsetneq (k+1)_u \uparrow$$

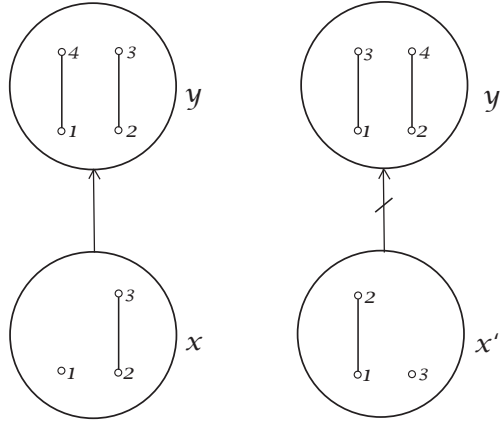
In either case, $y < u$ so $<$ is a partial order relation. To show that $<$ is a total order relation, suppose that $x \rightarrow y, z$. If $j_y \uparrow = j_z \uparrow$ for $j = 1, 2, \dots, |y|$, then the adjoined maximal element of y has the same parents as the adjoined maximal element of z . Hence, y and z are isomorphic ℓ -causets so $y = z$. Otherwise, by Lemma 3.3, $1_y \uparrow = 1_z \uparrow, \dots, j_y \uparrow = j_z \uparrow$ and $(j+1)_y \uparrow \subsetneq (j+1)_z \uparrow$ or $(j+1)_z \uparrow \subsetneq (j+1)_y \uparrow$ so $y < z$ or $z < y$. \square

Two elements a, b of an ℓ -causet are *comparable* if $a \leq b$ or $b \leq a$. Otherwise, a and b are *incomparable*. An *antichain* in $x \in \mathcal{P}'$ is a set of mutually incomparable elements of x . It is shown in [3] that the number of offspring $o(x)$ of $x \in \mathcal{P}'$ is the number of distinct antichains in x . Let $y_1 < y_2 < \dots < y_{o(x)}$ be the offspring of $x \in \mathcal{P}'$ ordered lexicographically. We call j the *succession* of y_j and $|y_i|$ the *generation* of y_j . If $x \in \mathcal{P}'$ with $x \neq \emptyset$, then x has a unique producer so its succession $s(x)$ is well defined. By convention we define $s(\emptyset) = 0$. We can uniquely specify each $x \in \mathcal{P}'$ by listing its *succession sequence*

$$(s(x_0), s(x_1), \dots, s(x_n))$$

where $x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_n = x$, $n = |x|$. Of course $x_0 = \emptyset$ and $s(x_0) = 0$, $s(x_1) = 1$ for all $x \in \mathcal{P}'$.

Example 1



If $x, y \in \mathcal{P}'$ we write $x \sim y$ if x and y are order isomorphic. Then \sim is an equivalence relation and we denote the equivalence class containing x by $[x]$. Let

$$\mathcal{Q} = \mathcal{P}' / \sim = \{[x] : x \in \mathcal{P}'\}$$

If $x \in \mathcal{P}'$, let $\hat{x} \in \mathcal{P}$ be x without the labels. Since we identify isomorphic causets we have $x \sim y$ if and only if $\hat{x} = \hat{y}$. Letting $\phi: \mathcal{Q} \rightarrow \mathcal{P}$ be $\phi([x]) = \hat{x}$

we see that ϕ is well-defined and it is easy to check that ϕ is a bijection. If $x, y \in \mathcal{P}'$ and $x \rightarrow y$, then clearly $\widehat{x} \rightarrow \widehat{y}$ in \mathcal{P} . Unfortunately, if $x \rightarrow y$ in \mathcal{P}' and $x' \sim x, y' \sim y$, then we can have $x' \not\rightarrow y'$ as Example 1 shows.

For $x, y \in \mathcal{P}'$ we write $[x] \rightarrow [y]$ if there exist $x', y' \in \mathcal{P}'$ with $x' \sim x, y' \sim y$ and $x' \rightarrow y'$. However, the identification $\phi([x]) = \widehat{x}$ is not very useful because $[x] \rightarrow [y]$ implies $\phi([x]) \rightarrow \phi([y])$ but $\phi([x]) \rightarrow \phi([y])$ need not imply $x \rightarrow y$ as the previous example shows. It is more useful to define a map $\psi: \Omega'_n \rightarrow \Omega_n$ as follows. If $\omega = \omega_1 \omega_2 \cdots \omega_n \in \Omega'_n$ we define

$$\psi(\omega) = \widehat{\omega} = \widehat{\omega}_1 \widehat{\omega}_2 \cdots \widehat{\omega}_n$$

Since any $x \in \mathcal{P}$ can be labeled, we have that $\psi: \Omega'_n \rightarrow \Omega_n$ is surjective.

Lemma 3.5. *Let $\omega = \omega_1 \omega_2 \cdots \omega_n, \omega' = \omega'_1 \omega'_2 \cdots \omega'_n \in \Omega'_n$. Then $\psi(\omega) = \psi(\omega')$ if and only if $\omega_j \sim \omega'_j, j = 1, 2, \dots, n$.*

Proof. If $\psi(\omega) = \psi(\omega')$ then $\widehat{\omega} = \widehat{\omega}'$ so

$$\widehat{\omega}_1 \widehat{\omega}_2 \cdots \widehat{\omega}_n = \widehat{\omega}'_1 \widehat{\omega}'_2 \cdots \widehat{\omega}'_n$$

which implies that $\widehat{\omega}_j = \widehat{\omega}'_j, j = 1, \dots, n$. We conclude that $\omega_j \sim \omega'_j, j = 1, \dots, n$. Conversely, if $\omega_j \sim \omega'_j, j = 1, \dots, n$, then $\widehat{\omega}_j = \widehat{\omega}'_j$. Hence

$$\psi(\omega) = \widehat{\omega}_1 \widehat{\omega}_2 \cdots \widehat{\omega}_n = \widehat{\omega}'_1 \widehat{\omega}'_2 \cdots \widehat{\omega}'_n = \psi(\omega') \quad \square$$

Let $x, y \in \mathcal{P}'$ with $x \rightarrow y'$ and $y' \sim y$ we write $y \sim_x y'$. Then \sim_x is an equivalence relation and we denote the equivalence classes by $[y]_x$. If $x \rightarrow y$, the *multiplicity* of $x \rightarrow y$ is denoted by $m(x \rightarrow y)$ and is defined by $m(x \rightarrow y) = |[y]_x|$. The *multiplicity* $m(\omega)$ of $\omega \in \Omega'_n$ is defined by $m(\omega) = |\psi^{-1}(\widehat{\omega})|$.

Lemma 3.6. *If $\omega = \omega_1 \omega_2 \cdots \omega_n \in \Omega'_n$, then*

$$m(\omega) = m(\omega_1 \rightarrow \omega_2) m(\omega_2 \rightarrow \omega_3) \cdots m(\omega_{n-1} \rightarrow \omega_n)$$

Proof. If $\omega' = \omega'_1 \omega'_2 \cdots \omega'_n \in \Omega'_n$ where $\omega'_i \sim \omega_i, i = 1, \dots, n$, then $\psi(\omega') = \psi(\omega)$. The number of n -paths of the form ω' is

$$m(\omega_1 \rightarrow \omega_2) m(\omega_2 \rightarrow \omega_3) \cdots m(\omega_{n-1} \rightarrow \omega_n)$$

The result follows. \square

4 Amplitude Processes on \mathcal{P}'

In this section we present examples of various APs on \mathcal{P}' . As shown in Section 2, these APs can be used to construct QSGPs on \mathcal{P}' . Moreover, we shall show in Section 5 that any QSGP on \mathcal{P}' can be “compressed” to a QSGP on \mathcal{P} . Recall that a transition amplitude on \mathcal{P}' is a map $\tilde{a}: \mathcal{P}' \times \mathcal{P}' \rightarrow \mathbb{C}$ such that $\tilde{a}(x, y) = 0$ if $x \not\rightarrow y$ and $\sum_y \tilde{a}(x, y) = 1$. We say that \tilde{a} is *covariant* if $\tilde{a}(x, y)$ is independent of the labeling of x and y ; that is, if $x \sim x'$, $y \sim y'$ then $\tilde{a}(x, y) = \tilde{a}(x', y')$

If $y = x \uparrow a$, then y is a *leaf offspring* of x if a has no more than one parent. It is clear that for $x \in \mathcal{P}'$, x has $|x| + 1$ leaf offspring and that there are $n!$ leaf offspring in \mathcal{P}'_n . For each $x \in \mathcal{P}'$, label the leaf offspring of x lexicographically, $y_1, y_2, \dots, y_{|x|+1}$. Define $\tilde{a}: \mathcal{P}' \times \mathcal{P}' \rightarrow \mathbb{C}$ by $\tilde{a}(x, y) = 0$ unless y is a leaf offspring of x and then

$$\tilde{a}(x, x_j) = -e^{2\pi i j(|x|+2)}, \quad j = 1, \dots, |x| + 1$$

where of course $i = \sqrt{-1}$. It is easy to check that \tilde{a} is a transition amplitude on \mathcal{P}' but \tilde{a} is not covariant.

If $\omega = \omega_1 \omega_2 \cdots \omega_n \in \Omega'_n$, then

$$a_n(\omega) = \tilde{a}(\omega_1, \omega_2) \cdots \tilde{a}(\omega_{n-1}, \omega_n)$$

Hence, if $\omega_1 \omega_2 \cdots \in \Omega'$ and at least one ω_j is not a leaf offspring, then

$$\mu_n(\{\omega\}^n) = |a_n(\omega_1 \cdots \omega_n)|^2 = 0$$

for n sufficiently large. Hence, $\{\omega\} \in \mathcal{S}(\Omega')$ and $\mu(\{\omega\}) = 0$. If every ω_n is a leaf offspring, then $\mu_n(\{\omega\}^n) = 1$ for all n so $\{\omega\} \in \mathcal{S}(\Omega')$ with $\mu(\{\omega\}) = 1$. As another example, let $A \subseteq \Omega'$ be the set of paths for which no first succession leaf offspring except ω_1 appear. Letting F be the set of

all first succession leaf offspring (except ω_1), we have

$$\begin{aligned}
\mu_n(A^n) &= \left| \sum \{ \tilde{a}(\omega_1, \omega_2) \cdots \tilde{a}(\omega_{n-1}, \omega_n) : \omega_j \notin F \} \right|^2 \\
&= \left| \sum \{ \tilde{a}(\omega_1, \omega_2) \cdots \tilde{a}(\omega_{n-2}, \omega_{n-1})(1 + e^{2\pi i/(n+1)}) : \omega_j \notin F \} \right|^2 \\
&= \left| \sum \{ \tilde{a}(\omega_1, \omega_2) \cdots \tilde{a}(\omega_{n-3}, \omega_{n-2})(1 + e^{2\pi i/n})(1 + e^{2\pi i/(n+1)}) : \omega_j \notin F \} \right|^2 \\
&\vdots \\
&= |(1 + e^{2\pi i/3})|^2 |(1 + e^{2\pi i/4})|^2 \cdots |(1 + e^{2\pi i/(n+1)})|^2 \\
&= (2 + 2 \cos 2\pi/3)^2 (2 + 2 \cos \pi/2)^2 \cdots (2 + 2 \cos 2\pi/(n+1))^2
\end{aligned}$$

We conclude that $\lim \mu_n(A^n) = \infty$ so $A \notin \mathcal{S}(\Omega)$.

As another example, let $\tilde{a}(x, y) = 0$ unless y is a leaf offspring of x and then $\tilde{a}(x, y) = (|x| + 1)^{-1}$. Then \tilde{a} is a covariant transition amplitude. Let $L_n \subseteq \Omega'_n$ be the set of n -paths $\omega = \omega_1 \omega_2 \cdots \omega_n$ such that ω_j are leaf offspring, $j = 1, 2, \dots, n$. Then $a_n(\omega) = 0$ for every $\omega \in \Omega'_n \setminus L_n$ and $a_n(\omega) = 1/n!$ for every $\omega \in L_n$. We conclude that $D_n(\omega, \omega') = 0$ unless $\omega, \omega' \in L_n$ in which case $D_n(\omega, \omega') = (n!)^{-2}$. Moreover, for $A, B \subseteq \Omega'_n$ we have

$$D_n(A, B) = \sum \{ D_n(\omega, \omega') : \omega \in A, \omega' \in B \} = \frac{|A \cap L_n| |B \cap L_n|}{(n!)^2}$$

and $\mu_n(A) = (n!)^{-2} |A \cap L_n|^2$. Let $L \subseteq \Omega'$ be the set of paths $\omega = \omega_1 \omega_2 \cdots$ where ω_j are leaf offspring, $j = 1, 2, \dots$. If $\omega \in \Omega' \setminus L$, then clearly $\mu_n(|\omega|^n) = 0$ so $\{\omega\} \in \mathcal{S}(\Omega')$ with $\mu(\{\omega\}) = 0$. If $\omega \in L$ then $\mu_n(\{\omega\}^n) = (n!)^{-2} \rightarrow 0$ so again $\{\omega\} \in \mathcal{S}(\Omega')$ with $\mu(\{\omega\}) = 0$. Also, if $\omega \in L$, then

$$\mu_n(\{\omega\}^n) = \frac{(n! - 1)^2}{(n!)^2} = \left(1 - \frac{1}{n!}\right)^2 \rightarrow 1$$

so $\{\omega\}' \in \mathcal{S}(\Omega)$ with $\mu(\{\omega\}') = 1$. In a similar way, if $\omega \in \Omega' \setminus L$ then $\{\omega\}' \in \mathcal{S}(\Omega)$ with $\mu(\{\omega\}') = 1$. Let $A \subseteq L$ be the subset of L consisting of paths $\omega = \omega_1 \omega_2 \cdots$ where each ω_i is a connected graph. It is easy to check that $|A^n| = (n-1)!$. Hence,

$$\mu_n(A^n) = \frac{[(n-1)!]^2}{(n!)^2} = \frac{1}{n^2} \rightarrow 0$$

Thus, $A \in \mathcal{S}(\Omega')$ and $\mu(A) = 0$. Since μ_n is the square of a measure on $\mathcal{C}(\Omega'_n)$, μ has a unique extension ν from $\mathcal{C}(\Omega')$ to \mathcal{A}' as a square of a measure. It follows that $\mathcal{S}(\Omega') = \mathcal{A}'$. However, $\mu(A) \neq \nu(A)$ for all $A \in \mathcal{A}'$, in general, because we need not have $A = \cap \text{cyl}(A^n)$.

We now briefly mention two covariant transition amplitudes that may have physical relevance. The first is complex percolation $\tilde{a}: \mathcal{P}' \times \mathcal{P}' \rightarrow \mathbb{C}$ [1]. As usual $\tilde{a}(x, y) = 0$ if $x \not\rightarrow y$. Let $p \in \mathbb{C}$ be arbitrary. If $y = x \uparrow a$ we define $\tilde{a}(x, y) = p^\pi (1 - p)^u$ where π is the number of parents of a and u is the number of elements of x that are incomparable to a . It is clear that \tilde{a} is covariant. To show that the Markov condition $\sum_y \tilde{a}(x, y) = 1$ holds, it is well-known from the classical theory that this condition holds if $0 \leq p \leq 1$. By analytic continuation, the condition still holds for complex p .

Our last example is a quantum action dynamics presented in [3]. This dynamics has the form of a discrete Feynman integral. Since this formalism was treated in detail in [3], we refer the reader to that reference for further consideration.

5 Compressing a QSGP from \mathcal{P}' to \mathcal{P}

This section shows that an arbitrary QSGP on \mathcal{P}' can be compressed in a natural way to a QSGP on \mathcal{P} . The *compression operator* $S_n: H'_n \rightarrow H_n$ is the linear operator defined by

$$S_n \chi_\omega = \chi_{\psi(\omega)} = \chi_{\hat{\omega}}$$

We also define the *covariance operator* $T_n: H_n \rightarrow H'_n$ as the linear operator given by

$$T_n \chi_\gamma = \sum \{ \chi_\omega \in H'_n : \hat{\omega} = \gamma \}$$

Lemma 5.1. *The operators S_n and T_n satisfy $T_n = S_n^*$.*

Proof. For $\omega \in \Omega'_n$ and $\gamma \in \Omega_n$ we have that

$$\langle S_n \chi_\omega, \chi_\gamma \rangle = \langle \chi_{\hat{\omega}}, \chi_\gamma \rangle = \delta_{\hat{\omega}, \gamma}$$

Moreover,

$$\langle \chi_\omega, T_n \chi_\gamma \rangle = \left\langle \chi_\omega, \sum \{ \chi_\omega \in H'_n : \hat{\omega} = \gamma \} \right\rangle = \delta_{\hat{\omega}, \gamma}$$

The result now follows. □

The next theorem is the main result of this section.

Theorem 5.2. *If $\{\rho_n\}$ is a QSGP for \mathcal{P}' , then $\widehat{\rho}_n = S_n \rho_n S_n^*$ is a QSGP for \mathcal{P} .*

Proof. We have that $\widehat{\rho}_n$ is positive because

$$\langle \widehat{\rho}_n f, f \rangle = \langle S_n \rho_n S_n^* f, f \rangle = \langle \rho_n S_n^* f, S_n^* f \rangle \geq 0$$

for every $f \in H_n$. The normalization condition follows from

$$\langle \widehat{\rho}_n 1_n, 1_n \rangle = \langle \rho_n S_n^* 1_n, S_n^* 1_n \rangle = \langle \rho_n T_n 1_n, T_n 1_n \rangle = \langle \rho_n 1_n, 1_n \rangle = 1$$

Consistency follows from

$$\begin{aligned} \langle \widehat{\rho}_{n+1} \chi_{(\omega \rightarrow)}, \chi_{(\omega' \rightarrow)} \rangle &= \langle \rho_{n+1} T_{n+1} \chi_{(\omega \rightarrow)}, T_{n+1} \chi_{(\omega' \rightarrow)} \rangle \\ &= \langle \rho_n T_n \chi_\omega, T_n \chi_{\omega'} \rangle = \langle \widehat{\rho}_n \chi_\omega, \chi_{\omega'} \rangle \end{aligned}$$

because

$$\langle \widehat{\rho}_{n+1} \chi_{A \rightarrow}, \chi_{B \rightarrow} \rangle = \langle \widehat{\rho}_n \chi_A, \chi_B \rangle$$

results from bilinearity. \square

The compression procedure is not reversible in the sense that if ρ_n is a QSGP for \mathcal{P} , then $\rho'_n = S_n^* \rho_n S_n$ need not be a QSGP for \mathcal{P}' . For example, in general

$$\langle \rho'_n 1_n, 1_n \rangle = \langle \rho_n S_n 1_n, S_n 1_n \rangle \neq 1$$

We now show that the compression operators grow according to multiplicity. First $S_n^* S_n: H'_n \rightarrow H'_n$ satisfies

$$S_n^* S_n \chi_\omega = T_n \chi_{\widehat{\omega}} = \sum \{ \chi_{\omega'} \in H'_n : \widehat{\omega}' = \widehat{\omega} \}$$

and $S_n S_n^*: H_n \rightarrow H_n$ satisfies

$$S_n S_n^* \chi_\gamma = S_n T_n \chi_\gamma = S_n \sum \{ \chi_\omega \in H'_n : \widehat{\omega} = \gamma \} = m(\gamma) \chi_\gamma$$

We thus see that $S_n S_n^*$ is diagonal with eigenvalues $\{m(\gamma) : \gamma \in \Omega_n\}$. Hence,

$$\|S_n S_n^*\| = \max \{m(\gamma) : \gamma \in \Omega_n\}$$

By the C^* -identity, we have

$$\|T_n\| = \|S_n\| = [\|S_n S_n^*\|]^{1/2} = [\max\{m(\gamma) : \gamma \in \Omega_n\}]^{1/2}$$

Suppose we have an AP $\{a_n\}$ on \mathcal{P}' . We have seen that $\rho_n = |a_n\rangle\langle a_n|$ is a QSGP for \mathcal{P}' . The corresponding QSGP $\hat{\rho}_n$ for \mathcal{P} becomes

$$\hat{\rho}_n = S_n |a_n\rangle\langle a_n| S_n^*$$

Hence,

$$\begin{aligned} \hat{\rho}_n \chi_\gamma &= S_n |a_n\rangle\langle a_n| T_n \chi_\gamma = S_n |a_n\rangle\langle a_n, T_n \chi_\gamma\rangle \\ &= S_n |a_n\rangle\left\langle a_n, \sum\{\chi_\omega \in H'_n : \hat{\omega} = \gamma\}\right\rangle \\ &= S_n |a_n\rangle \sum\{\langle a_n, \chi_\omega\rangle : \hat{\omega} = \gamma\} \\ &= S_n |a_n\rangle \sum\{\bar{a}_n(\omega) : \hat{\omega} = \gamma\} \end{aligned}$$

We conclude that

$$\begin{aligned} D_n(\gamma, \gamma') &= \langle \hat{\rho}_n \chi_{\gamma'}, \chi_\gamma \rangle = \langle S_n | a_n \rangle \langle a_n | T_n \chi_{\gamma'}, \chi_\gamma \rangle \\ &= \overline{\langle a_n, T_n \chi_{\gamma'} \rangle} \langle a_n, T_n \chi_\gamma \rangle \\ &= \sum\{\overline{a_n(\omega)} : \hat{\omega} = \gamma'\} \sum\{a_n(\omega') : \hat{\omega}' = \gamma\} \end{aligned}$$

It follows that

$$\mu_n(A) = \left| \sum_{\gamma \in A} \{a_n(\omega) : \hat{\omega} = \gamma\} \right|^2$$

and in particular,

$$\mu_n(\gamma) = \left| \sum \{a_n(\omega) : \hat{\omega} = \gamma\} \right|^2$$

We say that an AP $\{a_n\}$ on \mathcal{P}' is *covariant* if $a_n(\omega)$ is independent of the labeling of ω ; that is, $a_n(\omega) = a_n(\omega')$ whenever $\omega \sim \omega' (\hat{\omega} = \hat{\omega}')$. Of course, for a covariant transition amplitude \tilde{a} , the corresponding AP $\{a_n\}$ is covariant. If the AP $\{a_n\}$ on \mathcal{P}' is covariant, then

$$D_n(\gamma, \gamma') = m(\omega)m(\omega')\bar{a}_n(\omega)a_n(\omega')$$

where $\widehat{\omega} = \gamma$, $\widehat{\omega}' = \gamma'$. We also have

$$\mu_n(A) = \left| \sum_{\gamma \in A} \{m(\omega)a_n(\omega) : \widehat{\omega} = \gamma\} \right|^2$$

and in particular,

$$\mu_n(\gamma) = |m(\omega)|^2 |a_n(\omega)|^2$$

where $\widehat{\omega} = \gamma$. These last three equations exhibit the “explosion” in values resulting from multiplicity most clearly.

We can illustrate the situation directly for covariant APs as follows. Let $\widetilde{a}: \mathcal{P}' \times \mathcal{P}' \rightarrow \mathbb{C}$ be a covariant transitional amplitude. For $\gamma = \gamma_1\gamma_2 \cdots \gamma_n \in \Omega_n$, define

$$\widehat{a}_n(\gamma) = m(\omega_1 \rightarrow \omega_2)\widetilde{a}(\omega_1, \omega_2) \cdots m(\omega_{n-1} \rightarrow \omega_n)\widetilde{a}(\omega_{n-1}, \omega_n)$$

where $\widehat{\omega}_i = \gamma_i$, $i = 1, \dots, n$. Since \widetilde{a} is covariant, $\widehat{a}_n(\gamma)$ is well-defined and letting $\omega = \omega_1\omega_2 \cdots \omega_n \in \Omega'_n$ where $\widehat{\omega}_i = \gamma_i$, $i = 1, \dots, n$ we have $\widehat{\omega} = \gamma$ and

$$\widehat{a}_n(\gamma) = m(\omega)a_n(\omega)$$

Our previous results now follow.

6 Metrics

We have seen in Section 3 that any $x \in \mathcal{P}'_n$ is uniquely determined by its succession sequence

$$s(x) = (s_0, s_1, \dots, s_n)$$

If $y \in \mathcal{P}'_n$ has succession sequence

$$s(y) = (t_0, t_1, \dots, t_n)$$

we write $x \prec y$ if $s(x)$ precedes $s(y)$ lexicographically:

$$s_0 = t_0, s_1 = t_1, \dots, s_j = t_j, s_{j+1} < t_{j+1}$$

for some j with $j \leq n-1$. Then \prec is a total order on \mathcal{P}'_n . We can then well-order the elements of \mathcal{P}'_n as

$$x_1 \prec x_2 \prec \cdots \prec x_m$$

where $m = |\mathcal{P}'_n|$. We now define $\rho(x_i, x_j) = |i - j|$ for all $x_i, x_j \in \mathcal{P}'_n$. It is clear that $\rho(x_i, x_j) = 0$ if and only if $x_i = x_j$ and that $\rho(x_i, x_j) = \rho(x_j, x_i)$ for all $x_i, x_j \in \mathcal{P}'_n$. Moreover, by the triangle inequality for real numbers we have

$$\rho(x_i, x_j) = |i - j| \leq |i - k| + |k - j| = \rho(x_i, x_k) + \rho(x_k, x_j)$$

for all $x_k \in \mathcal{P}'_n$. Thus, ρ is a metric on \mathcal{P}'_n . For $x, y \in \mathcal{P}_n$ define

$$\rho(x, y) = \max \{ \rho(x', y') : \hat{x}' = x, \hat{y}' = y \}$$

if $x \neq y$ and define $\rho(x, y) = 0$, otherwise.

Theorem 6.1. *The map $\rho: \mathcal{P}_n \times \mathcal{P}_n \rightarrow \mathbb{R}$ is a metric on \mathcal{P}_n .*

Proof. Clearly, $\rho(x, y) \geq 0$ and $\rho(x, y) = \rho(y, x)$ for all $x, y \in \mathcal{P}_n$. Also, $\rho(x, x) = 0$. Suppose $x, y \in \mathcal{P}_n$ with $x \neq y$. If $\hat{x}' = x, \hat{y}' = y$, then $x' \neq y'$ so $\rho(x', y') > 0$. It follows that $\rho(x, y) > 0$. For $x, y, z \in \mathcal{P}_n$, if $x = y$ we have that

$$\rho(x, y) = 0 \leq \rho(x, z) + \rho(z, y)$$

so suppose that $x \neq y$. If $z = x$, then

$$\rho(x, z) + \rho(z, y) = \rho(x, y)$$

and a similar result holds if $z = y$. Finally, suppose that x, y, z are all distinct. Now there exists $x', y', z' \in \mathcal{P}'_n$ with $\hat{x}' = x, \hat{y}' = y, \hat{z}' = z$ and $\rho(x, y) = \rho(x', y')$. Since ρ is a metric on \mathcal{P}'_n we have that

$$\rho(x, y) = \rho(x', y') \leq \rho(x', z') + \rho(z', y') \leq \rho(x, z) + \rho(z, y)$$

Hence, the triangle equality holds so ρ is a metric on \mathcal{P}_n . \square

We now define metrics on Ω'_n and Ω_n . For $\omega, \omega' \in \Omega'_n$ with $\omega = \omega_2 \omega_2 \cdots \omega_n$, $\omega' = \omega'_1 \omega'_2 \cdots \omega'_n$ define

$$\rho(\omega, \omega') = \max \{ \rho(\omega_i, \omega'_i) : i = 1, 2, \dots, n \} \quad (6.1)$$

Theorem 6.2. *The map $\rho: \Omega'_n \times \Omega'_n \rightarrow \mathbb{R}$ is a metric.*

Proof. Clearly, $\rho(\omega, \omega') \geq 0$, $\rho(\omega, \omega') = \rho(\omega', \omega)$ and $\rho(\omega, \omega') = 0$ if and only if $\omega = \omega'$. If $\omega'' = \omega''_1 \omega''_2 \cdots \omega''_n \in \Omega'_n$ we have that $\rho(\omega, \omega') = \rho(\omega_j, \omega'_j)$ for some $j \in \{1, \dots, n\}$ and that

$$\rho(\omega, \omega') = \rho(\omega_j, \omega'_j) \leq \rho(\omega_j, \omega''_j) + \rho(\omega''_j, \omega'_j) \leq \rho(\omega, \omega'') + \rho(\omega'', \omega') \quad \square$$

We can define a metric on Ω_n in a similar way using (6.1). Other metrics on Ω'_n and Ω_n that might be convenient are

$$\rho_1(\omega, \omega') = \sum_{i=1}^n \rho(\omega_i, \omega'_i)$$

$$\rho_2(\omega, \omega') = \left[\sum_{i=1}^n \rho(\omega_i, \omega'_i)^2 \right]^{1/2}$$

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